Transfer Functions for Nonlinear Systems via Fourier-Borel Transforms

Sümer Can*
Aynur Ünal†

1 Abstract

An analytical functional can be expressed as a sum of some nonlinear functional expansions which we shall call Fliess's generalised expansions [2]. These nonlinear functional expansions are analogous to Fourier series or integral expansions of response functions of linear systems. The shuffle product which is the characteristic of the noncommutative algebra introduced plays a very significant role in this approach as explained in [1,2]. Moreover what makes this approach more attractive is the possibility of doing all of the noncommutative algebra on a computer in any of the currently available symbolic programming languages such as Macsyma, Reduce, PL1 and Lisp.

Nonlinear functional expansions for the solution of nonlinear ordinary differential equations can be summarised by the newly introduced Laplace-Borel transforms. Some properties of these transforms are obtained in [1] and [2]. Some further properties will be given in this paper for the first time.

The main theorem of the paper gives the transform of the response of the nonlinear system as a Cauchy product of its transfer function which is introduced for the first time here and the transform of the input function of the system together with memory effects.

Applications of this new transfer-function approach are given using nonlinear electronic circuits. Two categories of applications are presented, namely,

- analysis of circuits
- synthesis of circuits.

We would like to remind the redear that various other examples can be given from other nonlinear dynamical systems; for example nonlinear aerodynamics, nonlinear flight mechanics in which cases these two classes of problems can be called either direct problems or inverse problems.

*Department of Mechanical Engineering, Santa Clara University, Santa Clara, California, 95053.

†Department of Mechanical Engineering Santa Clara University, Santa Clara, California, 95053.

2 Introduction

The solution of dynamic problems by classical differential equation analysis is arduous, so that various methods of transform calculus have been developed to ease the burden and increase the understanding. It is interesting to note that such modern techniques stem from the work of 19th century mathematicians such as Fourier, Laplace, Cauchy, and others.

In this paper we shall develop a methodology to study nonlinear systems via transform methods. In particular we shall use the Laplace Borel transforms which are discussed in [1,2,3].

The dynamic performance of any initially dead system can be readily described by the frequency response function, $G(j\omega)$, thus:

$$G(j\omega) = \frac{\mathcal{F}_o}{\mathcal{F}_i} \tag{1}$$

where $\mathcal{F}$ denotes the Fourier transform. This notion is closely related to the transfer function, $G(s)$, where

$$G(s) = \frac{\mathcal{L}_o}{\mathcal{L}_i} \tag{2}$$

where $\mathcal{L}$ denotes the Laplace transform. The frequency response function and the transfer function are interchangeable by the substitution $s = j\omega$. Thus the Fourier transform of the system output, $F_0(j\omega)$ is given by

$$\mathcal{F}_o(j\omega) = G(j\omega).\mathcal{F}_i(j\omega) \tag{3}$$

where $\mathcal{F}_i(j\omega)$ is the input to the system expressed as a function of frequency either by the Fourier series for periodic functions or by the Fourier Integral for aperiodic functions. The Fourier transform enables a system response to transient excitations to be evaluated in terms of steady-state responses to sinusoidal excitations. Fourier methods have direct application to a few problems which are less easily solved by the Laplace transform, e.g.

- Random problems, i.e. noise and telecommunications, in which the input function can best be expressed as a frequency spectrum, i.e. a Fourier integral; and

ISCAS'88

- Transformaion of functions which are non zero for negative t and are therefore not Laplace transformable.

To find a Fourier pair from a Laplace pair

- If the Laplace transform, $F(s)$, has poles on or to the right of the imaginary axis there is no Fourier transform, i.e. $F(s) = \frac{1}{s}$ or $F(s) = \frac{1}{s-a}$ have no Fourier equivalent.
- Substitute jw for s in $F(s)$ to give $F(jw)$.
- Note that in using this method $f(t)$ is zero for negative t.

The Laplace-Borel transforms can be summarised as operators which we can obtain from the Laplace transformations as follows

$$|sF(s)|_{s=z_0^{-1}} \qquad (4)$$

except that the algebra on the noncommutative variable z_0 is richer. We have another type of product called shuffle product (Le mélange) in addition to Cauchy product. It is the shuffle product which provides the mechanism for us to take care of the nonlinear terms. The shuffle product and some related properties are presented in [1]. The connection between the Laplace and Fourier transforms is analogous to the one in between the Laplace-Borel and Fourier-Borel transforms. We can generalise the Laplace-Borel transforms to Fourier-Borel transforms in the same way that Fourier transforms are generalised from Laplace transforms.

3 Transfer Functions for Nonlinear Systems

Consider the following class of nonlinear systems with polynomial nonlinearity described by

$$\sum_{i=1}^{n} a_i \frac{d^i}{dt^i} x(t) + \sum_{j=1}^{m} b_j x^j(t) = f(t) \qquad (5)$$

We define the operator $\coprod^n$ as the shuffle product which is defined by Ünal in [1] repeated n times and the transfer function is the transform of the response due to a unit step function with zero initial conditions

$$\mathcal{G}(z_0, \overset{i}{\coprod}) \equiv X(z_0)\,|_{f(t)=u(t)} \qquad (6)$$

In the Laplace-Borel transform domain the following nonlinear differential equation becomes

$$\frac{dx(t)}{dt} + k_1 x(t) + k_2 x^2(t) = f(t) \qquad (7)$$

$$z_0\,[X(z_0) - x(0)] + k_1 X(z_0) + k_2\left[X(z_0) \coprod X(z_0)\right] = \mathcal{LB}\{f(t)\} \qquad (8)$$

From Ünal [1] we have

$$\mathcal{LB}[u(t)] = 1 \qquad (9)$$

with the zero initial conditions i.e. $x(0) = 0$ the transfer function for this nonlinear differential equation becomes

$$(z_0 + k_1 + k_2 \overset{2-1}{\coprod}) G(z_0, \overset{2-1}{\coprod}) = 1 \qquad (10)$$

or

$$G(z_0, \overset{2-1}{\coprod}) = \frac{1}{z_0 + k_1 + k_2 \coprod^{2-1}} \qquad (11)$$

Theorem 1 (Main Theorem) *The Laplace-Borel transform of the response of the nonlinear system considered is equal to the Cauchy product of the transfer function$[G(z_0, \coprod)]$ with the Laplace-Borel transform of the function which consists of the forcing function and the initial conditions of the response and all of its higher order derivatives.*

$$X(z_0) = G(z_0, \coprod).\mathcal{LB}\{f(t) + \sum_{k=0}^{i-1} a_i \frac{d^k}{dt^k} x(0) \frac{d^{i-k-1}}{dt^{i-k-1}} \delta(t)\} \qquad (12)$$

$$= G(z_0, \coprod).\mathcal{LB}\{f(t) + \sum_{k=0}^{i-1} a_i \frac{d^k}{dt^k} x(0) \frac{d^{i-k}}{dt^{i-k}} u(t)\} \qquad (13)$$

Proof:

Let us consider a nonlinear dynamical system described by an n^{th} order nonlinear differential equation with m^{th} order polynomial nonlinearity as follows :

$$\sum_{0}^{n} a_i \frac{d^i}{dt^i} x(t) + k_1 x(t) + \sum_{j=2}^{m} k_j x^j(t) = f(t) \qquad (14)$$

We shall demonstrate the proof on the sample problem and then consider the general form.

If the dynamical system has an evolution equation of the first order with quadratic nonlinearity i.e.

$$\frac{dx}{dt} + k_1 x(t) + k_2 x^2(t) = f(t) \qquad (15)$$

we want to express the Fourier(or Laplace)-Borel transform of the system in terms of its transfer function and the transform of the input function. To do this we shall take the Laplace-Borel transform of the given equation and hence we have

$$z_0 X(z_0) - x(0) + k_1 X(z_0) + k_2 X(z_0) \coprod X(z_0) = F(z_0) \qquad (16)$$

We defined the transfer function as the Laplace-Borel transform of the output of the system for a unit step function input with zero initial conditions (assuming that the system is initially dead).

$$f(t) = u(t) \qquad (17)$$

and

$$x(0) = 0 \tag{18}$$

hence

$$z_0X(z_0) + k_1X(z_0) + k_2X(z_0)\coprod^{2-1} = 1 \tag{19}$$

or

$$X(z_0) = \frac{1}{z_0 + k_1 + k_2\coprod} \tag{20}$$

$$= G(z_0, \coprod) \tag{21}$$

Now we go back to the original equation and take the Laplace-Borel transform of it as

$$z_0X(z_0) + k_1X(z_0) + k_2X(z_0)\coprod X(z_0) = F(z_0) + x(0) \tag{22}$$

$$\left(z_0 + k_1 + k_2\coprod\right)X(z_0) = [F(z_0) + x(0)] \tag{23}$$

or in terms of the transfer function

$$X(z_0) = G(z_0, \coprod)\,[F(z_0) + x(0)] \tag{24}$$

Notice that for the system of order one the memory effect consists of only the value of the response at the start.

We shall repeat this procedure for the more general case of a dynamical system described by an n^{th} order nonlinear ordinary differential equation with m^{th} order nonlinearity in it :

$$\sum_{i=1}^{n}\alpha_i\frac{d^i}{dt^i}x(t) + k_1x(t) + \sum_{j=2}^{m}k_jx^j(t) = f(t) \tag{25}$$

or in terms of the transfer function of the system we have:

$$X(z_0) = G(z_0, \coprod^{j-1}).\left[F(z_0) + \sum_{i=1}^{n}\alpha_i z_0^{i-1}\frac{d^{i-1}}{dx^{i-1}}x(0)\right] \tag{26}$$

Notice that for $i = 1$ and $j = 2$ and $\alpha_1 = 1$ the preceding general relation reduces to

$$X(z_0) = \left[z_0 + k_1 + k_2\coprod\right][F(z_0) + x(0)] \tag{27}$$

$$= G(z_0, \coprod^{2-1})\,[F(z_0) + x(0)] \tag{28}$$

Corollary 1 (Main Corollary) *For a memoryless system the Laplace-Borel transform of the output of the system is given by the Cauchy product of the transfer function with the Laplace-Borel transform of the input of the system.*

Proof of Corollary : The memory effects are lumped into the second term of the second factor of the Cauchy product i.e.

$$\sum_{i=1}^{n}\alpha_i z_0^{i-1}\frac{d^{i-1}}{dx^{i-1}}x(0) \tag{29}$$

which means function itself and its higher order derivatives evaluated at the zero time.

When the system has no memory we shall be left with

$$X(z_0) = G(z_0, \coprod)F(z_0) \tag{30}$$

In other words we can say that the main theorem has the same form as the linear systems if the nonlinear system has no memory.

4 fréchet Differential of the Response

Generating power series or equivalently Laplace-Borel transforms of the responses of nonlinear systems assume the existence of analyticity throughout the regime. However, we are fully aware of the fact that the loss of analyticity is very important and as explained in [4] is equivalent to the loss of Fréchet differentiability of the response and hence to the bifurcations of the response. Bifurcations in between regimes do take place at the critical values of the system's parameters and we can account for them by monitoring the Fréchet differential of the response.

To fix ideas, let us consider the nonlinear circuit problem :

$$\frac{d}{dt}v + k_1v + k_2v^2 = i(t) \tag{31}$$

where $i(t)$ represents the input current. Let us denote the Fréchet differential of the response by $\delta N[v, \eta]$ which is given by its definition as

$$\delta N[v, \eta] = \lim_{\epsilon \to 0}\left(\frac{v[i + \epsilon\eta] - v[i]}{\epsilon}\right) \tag{32}$$

or in Laplace-Borel transform domain,

$$\delta\mathcal{N}[V, \Omega] = \lim_{\epsilon \to 0}\left(\frac{V[i + \epsilon\eta] - V[i]}{\epsilon}\right) \tag{33}$$

where V, Ω are the Laplace-Borel transforms of v, η respectively. From the main theorem, we have

$$V[i] = G(z_0, \coprod).\,[\mathcal{LB}(i) + v(0)] \tag{34}$$

Similarly, using the main theorem once more we can write the output transformation for an input $i + \epsilon\eta$ as

$$V[i + \epsilon\eta] = G(z_0, \coprod).\,[\mathcal{LB}(i + \epsilon\eta) + v(0)] \tag{35}$$

since $\mathcal{LB}\ (i + \epsilon\eta) = \mathcal{LB}\ (i) + \epsilon\ \mathcal{LB}(\eta)$ we have

$$\delta\mathcal{N}[V, \Omega] = \lim_{\epsilon \to 0} G(z_0, \coprod)\mathcal{LB}(\eta) \tag{36}$$

If $\Omega(z_0) = \mathcal{LB}(\eta)$ then

$$\delta\mathcal{N}[V,\Omega] = \lim_{\epsilon \to 0} G(z_0, \coprod).\Omega(z_0) \quad (37)$$

or if we take $\eta(t) = u(t)$ i.e. unit step function, then $\Omega(z_0) = 1$ and the Fréchet differential becomes

$$\delta\mathcal{N}[V,\Omega] = G(z_0, \coprod).1 \quad (38)$$

or

$$\delta\mathcal{N}[V,\Omega] = G(z_0, \coprod) \quad (39)$$

which states that the Fréchet differential in Laplace-Borel transform domain is given by the transfer function of the system.

It is quite straightforward to generalise the previous result as

$$X(f) = G(z_0, \coprod).\left[F(z_0) + \sum_{i=1}^{n} \alpha_i z_0^{i-1} \frac{d^{i-1}}{dx^{i-1}} x(0)\right] \quad (40)$$

and

$$X(f+\epsilon\eta) = G(z_0, \coprod).\left[F(z_0) + \epsilon\Omega(z_0) + \sum_{i=1}^{n} \alpha_i z_0^{i-1} \frac{d^{i-1}}{dx^{i-1}} x(0)\right] \quad (41)$$

and hence

$$\delta\mathcal{N}[F,\Omega] = G(z_0, \coprod).\Omega(z_0) \quad (42)$$

or with $\eta(t) = u(t)$ i.e. unit step function

$$\delta\mathcal{N}[F,\Omega] = G(z_0, \coprod) \quad (43)$$

Hence we have the following theorem

Theorem 1 (Fréchet Differential) *The Fréchet differential of the response of a nonlinear dynamical system with polynomial type of nonlinearities is given in terms of the system's tranfer function and the variable of the Laplace-Borel transform as :*

$$\delta\mathcal{N}[X] = G(z_0, \coprod). \quad (44)$$

5 Concluding Remarks

We have demonstrated that the general response of nonlinear dynamical systems can be expressed in terms of their transfer functions in an analogous way to the linear systems.

We defined the transfer functions as the generalised series for the response of the nonlinear dynamical system which is initially at rest and which is loaded by a unit step function.

These transfer functions are obtainable through symbolic computer algebra and currently we have one of the following languages available to us: Macsyma , Reduce , PL1.

Analyticity of the response is assumed when the total response is expressed in terms of the transfer functions.

As a result of the last theorem on the Fréchet differential the loss of analyticity implis the loss of Fréchet differential [4], and the loss of Fréchet differential implies the loss of transfer function i.e. at Bifurcation points we can not determine the transfer functions.

6 Bibliography

[1] A. Ünal and M. Tobak, "An Algebraic Criterion for the Onset of Chaos in Nonlinear Dynamical Systems" NASA TM 89457 USA (August 1987).

[2] M. Fliess, M. Lamnahbi, F. Lamnahbi-Lagarrique, "An Algebraic Approach to Nonlinear Functional Expansions," IEEE Transactions on Circuits and Systems, 30, No. 8 (Aug. 1983), pp.554-570.

[3] M. Tobak, G.T. Chapman, A. Ünal, "Modeling Aerodynamic Discontinuities and the Onset of Chaos in Flight Dynamical Systems," NASA TM-89420 (Dec. 1986) .

[4] M. Tobak, A. Ünal, "Bifurcations in Unsteady Aerodynamics," NASA TM-88316 USA (June 1986).